hep-th/0407229

Noncommutative Tachyon Kinks as D$(p-1)$-branes from Unstable Dp-brane

Rabin Banerjee

S. N. Bose National Centre for Basic Sciences, Kolkata 700098, India
rabin@bose.res.in

Yoonbai Kim and O-Kab Kwon

BK21 Physics Research Division and Institute of Basic Science,
Sungkyunkwan University, Suwon 440-746, Korea
yoonbai@skku.edu okab@skku.edu

Abstract

We study noncommutative (NC) field theory of a real NC tachyon and NC U(1) gauge field, describing the dynamics of an unstable Dp-brane. For every given set of diagonal component of open string metric G_0, NC parameter θ_0, and interpolating electric field $\hat{E}$, we find all possible static NC kinks as exact solutions, in spite of complicated NC terms, which are classified by an array of NC kink-antikink and topological NC kinks. By computing their tensions and charges, those configurations are identified as an array of D0$\bar{\text{D}}$0 and single stable D0 from the unstable D1, respectively. When the interpolating electric field has critical value as $G_0^2 = \hat{E}^2$, the obtained topological kink becomes a BPS object with nonzero thickness and is identified as BPS D0 in the fluid of fundamental strings. Particularly in the scaling limit of infinite θ_0 and vanishing G_0 and $\hat{E}$, while keeping $G_0\theta_0 = \hat{E}\theta_0 = 1$, finiteness of the tension of NC kink corresponds to tensionless kink in ordinary effective field theory. An extension to stable D$(p-1)$ from unstable Dp is straightforward for pure electric cases with parallel NC parameter and interpolating two-form field.

1 Introduction

When the Neveu-Schwarz (NS) type background two-form field is turned on, the corresponding string theory contains two mass scales with a dimensionless string coupling g_s, i.e., they are the string scale $\sqrt{\alpha'}$ and the magnitude of the background field $1/\sqrt{|B|}$. Computation of propagator on a disc in terms of boundary conformal field theory (BCFT) gives a relation between closed and open string theory variables [1]. An intriguing aspect of Dirac-Born-Infeld (DBI) limit of open string theories, containing many derivative terms, is that the equivalence between the commutative spacetime theory in terms of closed string variables and the noncommutative (NC) spacetime analogue in terms of open string variables [2]. In the context of noncommutative field theory (NCFT) with fixed NC parameter θ, the NC scale $\sqrt{|\theta|}$ replaces the magnitude of the background field $1/\sqrt{|B|}$. In case of the pure magnetic background B_{ij}, two scales, θ^{ij} and α', in the NCFT can be decoupled [2]. On the other hand, with pure electric background B_{0i}, an (NC) field theory limit of the string theory is known to be not available in the limit of critical electric field or equivalently $|\theta^{0i}| \gg \alpha'$ limit due to possible problems like unitarity [3, 4]. In the context of string theory, an appropriate scaling limit toward this singular condition is known to lead to NC open string theories (NCOS) and a theory of light open membranes (OM) [4, 5].

Another noteworthy example in NCFT is existence of static soliton solutions, so-called GMS solitons [6], identified as D-branes and strings [7, 8]. Since almost all of the solitonic excitations are naturally codimension-two objects in $(2n+1)$-dimensions with spatial noncommutativity, vortex-like configurations are obtained in various NCFT's [9, 10], and they are used for the description of decay to D0-branes from D2$\bar{\text{D}}$2-system [11] through NC tachyon condensation [7, 8, 12].

It is well known that there also exist an unstable Dp-brane where p is odd for type IIA string theory and even for IIB. Its instability is represented by a real tachyonic degree, and it is an intriguing issue to obtain stable codimension-one D-brane from the unstable Dp-brane [13]. In the context of effective field theory (EFT) or NCFT, this question is translated as how to obtain a stable static kink solution where the tension of D$(p-1)$-brane is correctly computed [14]. In EFT, various static solitonic configurations of codimension-one including single kink and array of kink-antikink are obtained, which are thin or thick, with or without constant U(1) gauge field (equivalently NS-NS two form field) for arbitrary p [14, 15, 16, 17, 18]. Under a specific runaway tachyon potential, kinks identified by codimension-one D-branes are given as exact solutions [15, 16] and, with critical electric field, a thick topological kink is identified as a BPS soliton [16]. Up to the present, such a rich structure seems unlikely to be found in NCFT [19]. In this

paper, we will tackle this issue by combining the following two wisdoms: One is, in EFT, the action of real tachyon takes DBI type [20] and the other is, in EFT and NCFT, the NS-NS two-form field and U(1) gauge field are proven to share the same DBI type actions connected by Seiberg-Witten (SW) map [2].

We propose a new DBI type action of NC tachyon with coupling of NC U(1) gauge field, describing dynamics of an unstable Dp-brane. An important feature of this action is its equivalence (up to the first non-trivial order in the NC parameter) with the corresponding DBI action of tachyons in the commutative case. The equations of motion obtained from our NC DBI action, for a flat unstable D1-brane with arbitrary diagonal component of open string metric G_0 and interpolating electric field $\hat{E}$, lead to static kink solutions. We abstracted all such solutions as an array of NC kink-antikink and topological (NC) kinks. Furthermore, when the interpolating electric field has the critical value $\hat{E}^2 = G_0^2$, the topological kink reduces to a BPS object with non-vanishing thickness that is identified as a BPS D0 in the fluid of fundamental strings. Properties of the obtained kinks are consistent at on-shell level with those in terms of boundary string field theory [21], EFT [14, 15, 16, 17, 18], and BCFT [22]. For the rolling tachyon solutions in the presence of pure electric field, BCFT results are compared with those of NCFT at the level of classical energy-momentum tensor [23].

The rest of the paper is organized as follows. In section 2 we review briefly DBI action of NC U(1) gauge field and then propose an extension of this action such that it includes a NC tachyon field. For slowly varying fields, equivalence between ordinary effective action and NCFT action (in the presence of the tachyon) is shown up to leading order of the NC parameter; this is presented in Appendix A. In section 3, the (1+1)-dimensional case is considered in detail. All the static NC kink solutions are obtained as exact solutions. They are identified as codimension-one branes by computing their tensions and charges. We conclude in section 4 with a brief discussion.

2 DBI Gauge Field and NC Tachyon

In this section, we discuss our new proposal for the DBI type action of NC tachyon coupled to NC U(1) gauge field. First, in subsection 2.1, we review briefly both DBI action of U(1) gauge field and NS-NS two-form field in the background of closed string metric, and that of NC U(1) gauge field in the background of open string metric. Various relations due to equivalence between those two actions are also presented together with interpolating two-form field. Then, in subsection 2.2, we propose a DBI type NC action of a real NC tachyon coupled to NC U(1) gauge field based on the DBI type action of ordinary effective

field theory. Equivalence of the two tachyon actions is shown up to the leading order of NC parameter, for slowly varying fields.

2.1 Review on DBI action of NC U(1) gauge field

DBI action describes slowly varying gauge field $A_\mu(x)$ on a single Dp-brane;

$$S_{\rm DBI} = -\frac{1}{g_{\rm s}(2\pi)^{\frac{p-1}{2}}}\int d^{p+1}x\sqrt{-\det(g_{\mu\nu}+B_{\mu\nu}+F_{\mu\nu})}\,, \tag{2.1}$$

where string coupling $g_{\rm s}$, metric $g_{\mu\nu}$, and a constant NS-NS two-form field $B_{\mu\nu}$ are the closed string variables read on the Dp-brane. $F_{\mu\nu}$ denotes field strength tensor of the U(1) gauge field, whose value is a constant everywhere or vanishes at asymptotic region. In addition to equation of the gauge field, it satisfies Bianchi identity

$$\partial_\mu F_{\nu\rho}+\partial_\nu F_{\rho\mu}+\partial_\rho F_{\mu\nu}=0. \tag{2.2}$$

Note that tension of the Dp-brane $\mathcal{T}_p$ is inversely proportional to the closed string coupling

$$\mathcal{T}_p = \frac{1}{g_{\rm s}(2\pi)^{\frac{p-1}{2}}}. \tag{2.3}$$

A BCFT calculation of propagator on a disc, which corresponds to a point splitting regularization of string theory, provides open string metric $G^{\mu\nu}$ and NC parameter $\theta^{\mu\nu}$ in terms of the closed string variables as [1] (see also [2])

$$G_{\mu\nu} = g_{\mu\nu}-(Bg^{-1}B)_{\mu\nu}, \tag{2.4}$$

$$\theta^{\mu\nu} = -\left(\frac{1}{g+B}B\frac{1}{g-B}\right)^{\mu\nu}. \tag{2.5}$$

NC DBI type action, which is proven to be equivalent to ordinary DBI action (2.1) in the limit of slowly varying fields [2], is given by

$$\hat{S}_{\rm DBI} = -\frac{1}{G_{\rm s}(2\pi)^{\frac{p-1}{2}}}\int d^{p+1}x\sqrt{-\det(G_{\mu\nu}+\Phi_{\mu\nu}+\hat{F}_{\mu\nu})}\,, \tag{2.6}$$

where NC field strength tensor $\hat{F}_{\mu\nu}$ is defined by

$$\hat{F}_{\mu\nu}=\partial_\mu\hat{A}_\nu-\partial_\nu\hat{A}_\mu-i\hat{A}_\mu*\hat{A}_\nu+i\hat{A}_\nu*\hat{A}_\mu, \tag{2.7}$$

and $\Phi_{\mu\nu}$ is an interpolating two-form field depending on $g_{\mu\nu}$, $B_{\mu\nu}$, and $\theta_{\mu\nu}$ [2]. Star product for NC fields is defined by

$$f(x)*g(x)\equiv e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\zeta^\nu}}f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}. \tag{2.8}$$

NC Bianchi identity is the natural NC deformation of the ordinary one (2.2) [24]

$$\hat{D}_\mu \hat{F}_{\nu\rho} + \hat{D}_\nu \hat{F}_{\rho\mu} + \hat{D}_\rho \hat{F}_{\mu\nu} = 0, \tag{2.9}$$

where NC covariant derivative is

$$\hat{D}_\mu = \partial_\mu - i[\hat{A}_\mu,\ \ ]_*\,, \qquad [A,B]_* = A*B - B*A. \tag{2.10}$$

When the field strength tensor $F_{\mu\nu}$ and the interpolating field $\Phi_{\mu\nu}$ vanish, its NCFT analogue $\hat{F}_{\mu\nu}$ also vanishes so that the coupling of the open string $G_{\rm s}$ is expressed by the closed string theory variables

$$G_{\rm s} = g_{\rm s}\sqrt{\det(1+g^{-1}B)}\,. \tag{2.11}$$

Note that, from the coefficient of the quadratic term, electromagnetic coupling $g_{\rm EM}$ is also identified as

$$\frac{1}{g_{\rm EM}^2} = \frac{1}{G_{\rm s}(2\pi)^{\frac{p-1}{2}}} = \frac{1}{g_{\rm s}\sqrt{\det(1+g^{-1}B)}(2\pi)^{\frac{p-1}{2}}}. \tag{2.12}$$

For a given closed string metric with $\sqrt{-g} > 0$, reality condition of the actions Eqs. (2.1) and (2.6) is presumed. Therefore, validity of the open string variables, $G_{\mu\nu}$ (2.4), $\theta_{\mu\nu}$ (2.5), and $G_{\rm s}$ (2.11), is justified by the critical line characterized by $\sqrt{\det(1+g^{-1}B)}$ which is vanishing DBI Lagrange density $\mathcal{L}_{\rm DBI}/\sqrt{-g}$ or equivalently $G_{\rm s}/g_{\rm s}$ in Eq. (2.11). Since the DBI action (2.1) is valid in weak string coupling limit ($g_{\rm s} \to 0$), it means $G_{\rm s} \to 0$ so does $g_{\rm EM}$ (2.12). Simultaneously, the open string metric (2.4) vanishes, $\det G_{\mu\nu} \to 0$, and the NC parameter diverges, $\det \theta^{\mu\nu} \to \infty$. It seems that the NCFT of pure $G_{\mu\nu}$ of our interest becomes singular as the NS-NS field $B_{\mu\nu}$ approaches critical value, and physically meaningless for the NS-NS field $B_{\mu\nu}$ larger than the critical value.

2.2 NC tachyon action

Let us begin this subsection by introducing an effective tachyon action for the unstable Dp-brane system in ordinary spacetime [20]

$$S = -\frac{1}{g_{\rm s}(2\pi)^{\frac{p-1}{2}}}\int d^{p+1}x\; V(T)\sqrt{-\det(g_{\mu\nu}+B_{\mu\nu}+F_{\mu\nu}+\partial_\mu T\partial_\nu T)}\,. \tag{2.13}$$

Since tachyon potential $V(T)$ measures variable tension of the unstable D-brane, it can be any runaway potential connecting monotonically

$$V(T=0) = 1 \ \text{ and } \ V(T=\pm\infty) = 0. \tag{2.14}$$

Physics of tachyon condensation is largely irrelevant to the detailed form of the tachyon potential once it satisfies the runaway property and the boundary values (2.14) [25]. For example, both the basic runaway behavior of rolling tachyon solutions [26], existence of various tachyon kink solutions [16, 18], and their BPS nature with zero thickness [14] are attained irrespective of the specific shape of the potential, which just reflects a detailed decaying dynamics of the unstable Dp-brane. So will be the NC tachyon kinks in the context of NCFT, which will be shown in the next section.

Here we also adopt a specific form of the tachyon potential $V(T)$ [27, 28, 29] as

$$V(T) = \frac{1}{\cosh(T/R)}, \tag{2.15}$$

where $R = \sqrt{2}$ for superstring theory and 2 for bosonic string theory. This potential (2.15) has some nice features: (i) It is derived in open string theory by taking into account the fluctuations around $\frac{1}{2}$S-brane configuration with the higher derivatives neglected, i.e., $\partial^2 T = \partial^3 T = \cdots = 0$ [30]. (ii) Exact solutions are obtained for rolling tachyon [18, 28] and tachyon kink solutions on unstable Dp without or with a coupling of U(1) gauge field for arbitrary p [15, 16, 18]. (iii) Some of the obtained classical solutions $T(x)$ in the EFT (2.13), e.g., rolling tachyons and tachyon kinks, can be directly translated to BCFT tachyon profiles $\tau(x)$ in open string theory [15, 22] described by the following point transformation obtained in Ref. [30], [1]

$$\frac{\tau(x)}{R} = \sinh\left(\frac{T(x)}{R}\right). \tag{2.16}$$

(iv) The period of the obtained array solutions of tachyon kink-antikink [15, 16, 18] or tube-antitube [32, 33] is independent of the integration constant of the equation of motion only under this potential (2.15) [33], which is a crucial property in string theory if one wishes to identify the array solution as a configuration on a circle or a sphere of a fixed radius [22, 34]. We will also demonstrate that the properties (ii)–(iv) are indeed shared with the kinks in NCFT (see the next section).

When the tachyon is considered as a real NC scalar field in the context of NCFT, an action with quadratic kinetic term of it was proposed [7] and has been used for studying physics of unstable D-branes [7, 8, 11]. If we adopt such action for NC tachyon, then the relation between it and the ordinary tachyon action (2.13) may not be made clearly. In this paper, we propose another NC tachyon action based on Eq. (2.13)

$$\hat{S} = -\frac{\hat{\mathcal{T}}_p}{2}\int d^{p+1}x \left[\hat{V}(\hat{T}) * \sqrt{-\hat{X}_*} + \sqrt{-\hat{X}_*} * \hat{V}(\hat{T})\right]. \tag{2.17}$$

[1]Intriguingly, such a transformation (for $R = 1$) maps an Euclidean sphere having radius τ with the corresponding hyperbolic sphere having radius T [31].

Each term in Eq. (2.17) is obtained by replacing ordinary product with star product, *e.g.*, NC determinant in $(p+1)$-dimensional spacetime is defined as

$$\hat{X}_* = \det{}_* \hat{X}_{\mu\nu} \equiv \frac{1}{(p+1)!} \epsilon_{\mu_1\mu_2\cdots\mu_{p+1}} \epsilon_{\nu_1\nu_2\cdots\nu_{p+1}} \hat{X}_{\mu_1\nu_1} * \hat{X}_{\mu_2\nu_2} * \cdots * \hat{X}_{\mu_{p+1}\nu_{p+1}}, \tag{2.18}$$

and full symmetrization is implied,

$$\begin{aligned} \left[\hat{A}_1 \hat{A}_2 \cdots \hat{A}_n\right]_* &= \frac{1}{n!} \Big(\hat{A}_1 * \hat{A}_2 * \cdots * \hat{A}_n + \hat{A}_1 * \hat{A}_3 * \cdots * \hat{A}_n \\ &\qquad + \cdots (\text{all possible permutations}) \Big). \end{aligned} \tag{2.19}$$

To be specific, we have

$$\begin{aligned} \hat{V}(\hat{T}) &\equiv 1 - \frac{1}{2}\frac{\hat{T}}{R} * \frac{\hat{T}}{R} + \frac{5}{24}\frac{\hat{T}}{R} * \frac{\hat{T}}{R} * \frac{\hat{T}}{R} * \frac{\hat{T}}{R} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} \left[\left(\frac{\hat{T}}{R} \right)^{2k} \right]_* = \left[\frac{1}{\cosh(\hat{T}/R)} \right]_*, \end{aligned} \tag{2.20}$$

$$\hat{X}_* = \det{}_* \left[G_{\mu\nu} + \hat{F}_{\mu\nu} + \frac{1}{2} \left(\hat{D}_\mu \hat{T} * \hat{D}_\nu \hat{T} + \hat{D}_\nu \hat{T} * \hat{D}_\mu \hat{T} \right) \right], \tag{2.21}$$

where E_{2k} is the Euler number. A quantity

$$\hat{\mathcal{T}}_p \equiv \frac{1}{G_s (2\pi)^{\frac{p-1}{2}}} = \frac{\mathcal{T}_p}{\sqrt{\det(1 + g^{-1}B)}} \tag{2.22}$$

is introduced for convenient connection between the NC DBI action (2.17) and the ordinary DBI action (2.13) in the limit of vanishing tachyon, $\hat{T} = 0$.

The aforementioned procedure of star products between all the fields and their full symmetrization is not unique but seems likely to be a natural choice in this stage.[2] This kind of ambiguity is genuine even in usual NC scalar field theory with equal to or more than ϕ^6-potential term [35], and affects much on solitonic spectra, particularly on codimension-two objects [36]. However, note that the kink solutions, which will be dealt in the next section, are supported irrespective of such detailed procedure for obtaining the NC tachyon action once it takes a DBI type. For slowing varying $\hat{F}_{\mu\nu}$ and $\hat{D}_\mu \hat{T}$, the star products in $\hat{X}_*$ may be replaced by the ordinary products according to the approach of Ref. [2]. Then the NC DBI action (2.17) is simplified as

$$\hat{S} = -\hat{\mathcal{T}}_p \int d^{p+1}x \left[\hat{V}(\hat{T}) \sqrt{-\hat{X}} + \mathcal{O}(\partial \hat{F}, \partial \hat{D} \hat{T}) \right], \tag{2.23}$$

[2]This is a valid symmetrization since it is similar to the familiar Weyl ordering prescription which is compatible with the canonical definition of star product taken here.

where all the products in the determinant are ordinary products

$$\hat{X} = \det(G_{\mu\nu} + \hat{F}_{\mu\nu} + \hat{D}_\mu \hat{T} \hat{D}_\nu \hat{T}). \tag{2.24}$$

Now star products are contained only in the tachyon potential $\hat{V}(\hat{T})$. Since the DBI type effective action of tachyon can be valid only for slowly varying tachyon and U(1) gauge field, it is enough to prove up to leading order of the NC parameter, the equivalence with the corresponding DBI action in the commutative case. This would also establish consistency with [2] where an analogous demonstration was carried out for the non-tachyonic theory. Since it is lengthy, we give the detailed proof in Appendix A. The total derivative difference in Lagrange density arises from the fact that both the NC gauge field action and the NC tachyon action of our interest can be derived in string theory at on-shell, which are not sensitive to such total derivatives. The difference by $\mathcal{O}(\partial F)$ or $\mathcal{O}(\partial \hat{D} T)$ in the Lagrange density is expected from the beginning since such terms are neglected when the Lagrange densities (2.23) and (2.13) are derived for both NC and ordinary field theory cases.

Equations of motion for the NC tachyon and U(1) gauge field are given by

$$\hat{D}_\mu \left(\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}} \hat{C}_{\mathrm{S}}^{\mu\nu} \hat{D}_\nu \hat{T} \right) - \left[\frac{\sinh(\hat{T}/R)}{R\cosh^2(\hat{T}/R)} \right]_* \sqrt{-\hat{X}} = 0, \tag{2.25}$$

$$\hat{D}_\mu \left(\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}} \hat{C}_{\mathrm{A}}^{\mu\nu} \right) + i \left[\hat{T} \, , \, \frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}} \hat{C}_{\mathrm{S}}^{\mu\nu} \hat{D}_\nu \hat{T} \right]_* = 0, \tag{2.26}$$

where $\hat{C}_{\mathrm{S}}^{\mu\nu}$ and $\hat{C}_{\mathrm{A}}^{\mu\nu}$ are symmetric and antisymmetric parts of the cofactor $\hat{C}^{\mu\nu}$ of the matrix $(\hat{X})_{\mu\nu}$, computed as

$$\delta_{\hat{T}} \sqrt{-\hat{X}} = -\frac{\delta_{\hat{T}} \hat{X}}{2\sqrt{-\hat{X}}} = -\frac{\hat{C}^{\mu\nu}}{2\sqrt{-\hat{X}}} \delta_{\hat{T}} \hat{X}_{\mu\nu}. \tag{2.27}$$

The detailed procedure deriving the equations (2.25)–(2.26) from the NC action (2.17) is given in Appendix B.

NC version of the energy-momentum tensor is read by a systematic way [24] as follows

$$\hat{T}^{\mu\nu} \equiv \frac{2}{\sqrt{-G}} \frac{\delta \hat{S}}{\delta G_{\mu\nu}} = \frac{\hat{\mathcal{T}}_p \hat{V} \hat{C}_{\mathrm{S}}^{\mu\nu}}{\sqrt{-G}\sqrt{-\hat{X}}} \tag{2.28}$$

which is conserved covariantly

$$\hat{D}_\mu \hat{T}^{\mu\nu} = 0. \tag{2.29}$$

3 Codimension-one Branes from Unstable D1-brane

We consider a flat unstable D1-brane in NCFT having NC parameter θ_0 and diagonal component of open string metric G_0 with interpolating electric field $\hat{E}$. In this section, we find all the static solitonic objects of codimension-one, interpreted as an array of D0$\bar{\text{D}}$0 or D0-brane.

Suppose the closed string metric has only diagonal components and the antisymmetric tensor field has electric components as

$$(g_{\mu\nu}) = \begin{pmatrix} -g_0 & 0 \\ 0 & g_0 \end{pmatrix}, \tag{3.1}$$

$$(B_{\mu\nu}) = \begin{pmatrix} 0 & E_0 \\ -E_0 & 0 \end{pmatrix}, \tag{3.2}$$

where g_0 and E_0 are constants, and $g_0 = 1$ does not lose generality in our discussion. By comparing the actions for $p = 1$, Eq. (2.1) with $F_{\mu\nu} = 0$ and Eq. (2.6) with $\hat{F}_{\mu\nu} = 0$, we read the open string coupling $G_{\rm s}$ (2.11) (or equivalently $\hat{\mathcal{T}}_1$ in NCFT) and metric

$$(G_{\mu\nu}) = \begin{pmatrix} -G_0 & 0 \\ 0 & G_0 \end{pmatrix}. \tag{3.3}$$

Specifically, for nonnegative g_0 and $g_0^2 - E_0^2$, we naturally have

$$G_0 = \frac{g_0^2 - E_0^2}{g_0} \geq 0, \quad \frac{G_{\rm s}}{\sqrt{G_0}} = \frac{g_{\rm s}}{\sqrt{g_0}} \quad (\sqrt{G_0}\hat{\mathcal{T}}_1 = \sqrt{g_0}\mathcal{T}_1), \tag{3.4}$$

while the NC parameter has the form

$$(\theta^{\mu\nu}) = \begin{pmatrix} 0 & \theta_0 \\ -\theta_0 & 0 \end{pmatrix}, \qquad \theta_0 = \frac{E_0}{g_0^2 - E_0^2}. \tag{3.5}$$

We are interested in D0-branes from the unstable D1-brane, given as static solitonic configurations. Therefore, NC fields are assumed to depend only on x-coordinate, i.e., $\hat{T} = \hat{T}(x)$ and $\hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}(x)$. If we choose a Weyl gauge $\hat{A}_0 = 0$ for convenience, we have simplified expressions as follows

$$\begin{aligned} &\hat{F}_{01} \equiv \hat{E} = \partial_0 \hat{A}_1 - \partial_1 \hat{A}_0 - i[\hat{A}_0, \hat{A}_1] = \partial_0 \hat{A}_1, \\ &\hat{T} * \hat{T} = \hat{T}^2, \quad \hat{D}_\mu \hat{T} = \delta_{\mu 1} \hat{T}'(1 + \hat{E}\theta_0), \quad \hat{D}_\mu \hat{T} * \hat{D}_\nu \hat{T} = \delta_{\mu 1}\delta_{\nu 1} \hat{T}'^2 (1 + \hat{E}\theta_0)^2, \end{aligned} \tag{3.6}$$

where $\hat{T}' = d\hat{T}/dx$. Remarkably, under this gauge, every star product in the NC DBI action (2.17) and the NC equations of motion (2.25)–(2.26) is replaced by ordinary product.[3] In this sense, the codimension-one objects of our interest are insensitive to the way

[3]It appears therefore that the Weyl gauge "abelianizes" the NC theory just as the axial gauge (say $A_1 \approx 0$) "abelianizes" the usual Yang–Mills theory, allowing for a solution of the Gauss constraint.

how to attach the star product, as previously mentioned. This property (3.6) allows application of the same point transformation (2.16) to the NC tachyon field $\hat{T}$. Performing this transformation (2.16) to $\hat{T}$, we have $\hat{\tau}\left(= R\sinh(\hat{T}/R)\right)$ and an NC action

$$\hat{S} = -\hat{\mathcal{T}}_1 \int d^2x\, \hat{V}(\hat{\tau})\sqrt{-\hat{X}}, \tag{3.7}$$

where the tachyon potential (2.20) and the NC determinant (2.21) become

$$\hat{V}(\hat{\tau}) = \frac{1}{\sqrt{1+\hat{\tau}^2/R^2}}, \tag{3.8}$$

$$-\hat{X}_* = -\hat{X} = G_0^2 - \hat{E}^2 + G_0(1+\hat{E}\theta_0)^2\hat{V}^2\hat{\tau}'^2. \tag{3.9}$$

Conjugate momentum of the gauge field $\hat{\Pi}$ defined by

$$\hat{\Pi} \equiv \frac{\delta\hat{S}}{\delta\hat{F}_{01}} = \frac{\hat{\mathcal{T}}_1\hat{V}}{\sqrt{-\hat{X}}}\hat{E} \tag{3.10}$$

is time independent due to spatial component of the gauge field equation (2.26), i.e., $\partial_0\hat{\Pi} = 0$ under the Weyl gauge. Then the equations of motion for the tachyon (2.25) and Gauss' law constraint, time component of the gauge field equation (2.26), are

$$(1+\hat{E}\theta_0)\left[\frac{\hat{\mathcal{T}}_1\hat{V}}{\sqrt{-\hat{X}}}\hat{V}^2G_0(1+\hat{E}\theta_0)\hat{\tau}'\right]' + \frac{\hat{\mathcal{T}}_1\hat{V}}{\sqrt{-\hat{X}}}\frac{\hat{V}^2\hat{\tau}}{R^2}\left[-\hat{X} + G_0\hat{V}^2(1+\hat{E}\theta_0)^2\hat{\tau}'^2\right] = 0, \tag{3.11}$$

$$(1+\hat{E}\theta_0)(\hat{\Pi})' = (1+\hat{E}\theta_0)\left(\frac{\hat{\mathcal{T}}_1\hat{V}}{\sqrt{-\hat{X}}}\hat{E}\right)' = 0. \tag{3.12}$$

Instead of solving the complicated tachyon equation (3.11), it is convenient to consider x-component of the conservation of energy-momentum (2.29)

$$\hat{D}_1\hat{T}^{11} = (1+\hat{E}\theta_0)\left(\frac{-\hat{\mathcal{T}}_1\hat{V}}{\sqrt{-\hat{X}}}\right)' = 0. \tag{3.13}$$

When $1+\hat{E}\theta_0$ vanishes, the derivative terms of the tachyon field disappear as shown in Eq. (3.6) and thereby the equations of motion (3.11)–(3.13) become trivial. So no nontrivial solitonic object is obtained except for trivial vacuum solutions, $\hat{\tau} = 0$ or $\hat{\tau} = \pm\infty$.

When $1+\hat{E}\theta_0 \neq 0$, Eq. (3.13) dictates constancy of $\hat{T}^{11}$ as

$$-\hat{T}^{11} = \frac{\hat{\mathcal{T}}_1 \hat{V}}{\sqrt{-\hat{X}}}. \tag{3.14}$$

Then the gauge equation (3.12) allows only constant electric field, $\hat{E}$ = constant, and simultaneously it means that constancy of conjugate momentum to the gauge field, $\hat{\Pi} = -\hat{T}^{11}\hat{E}$ due to Eq. (3.10). Momentum density $\hat{T}^{01}$ vanishes, and then energy density is the only component of energy-momentum tensor with nontrivial profile

$$-\hat{T}^0_{\ 0} = \hat{\Pi}\frac{\hat{E}}{G_0} + \frac{\hat{E}/G_0}{\hat{\Pi}}(\hat{\mathcal{T}}_1\hat{V})^2. \tag{3.15}$$

Obviously these solutions satisfy the tachyon equation (3.11). Finally, Eq. (3.14) is rewritten as

$$\mathcal{E}_1 = \frac{1}{2}\hat{\tau}'^2 + U_1(\hat{\tau}), \tag{3.16}$$

where

$$\mathcal{E}_1 = \frac{1}{2}\left[\frac{\hat{\mathcal{T}}_1^2}{G_0(-\hat{T}^{11}(1+\hat{E}\theta_0))^2} - \frac{G_0^2-\hat{E}^2}{G_0(1+\hat{E}\theta_0)^2}\right], \tag{3.17}$$

$$U_1(\hat{\tau}) = \frac{1}{2}\hat{\omega}^2R^2\left(\frac{1}{\hat{V}^2}-1\right), \qquad \hat{\omega} = \frac{1}{R}\sqrt{\frac{G_0^2-\hat{E}^2}{G_0(1+\hat{E}\theta_0)^2}}. \tag{3.18}$$

For a given geometry with a fixed $g_0 \neq 0$ (or equivalently a fixed G_0) and the compactification scale R, the system of our interest seems to be classified by three parameters, i.e., they are negative pressure $-\hat{T}^{11}$, NC parameter θ_0, and NC electric field $\hat{E}$. On the other hand, the solution space of static codimension-one objects is classified by two parameters $\mathcal{E}_1$ and $\hat{\omega}$. Specifically, the following two combinations are read from Eqs. (3.17)–(3.18)

$$\frac{\hat{\mathcal{T}}_1^2}{G_0[-\hat{T}^{11}(1+\hat{E}\theta_0)]^2}, \qquad \frac{G_0^2-\hat{E}^2}{G_0(1+\hat{E}\theta_0)^2}. \tag{3.19}$$

In fact, value of the NC electric field $\hat{E}$ is constant version of two-form auxiliary field $\Phi_{\mu\nu}$ in Eq. (2.6). It interpolates between the limit of ordinary EFT with a constant electric field E for $(\theta_0 = 0, E = \hat{E})$ and that of NCFT with $E/g_0 = G_0\theta_0$ for $(\theta_0 \neq 0, \hat{E} = 0)$. Since the analogue of $\hat{\omega}$ in ordinary effective theory is $\omega = \sqrt{(g_0^2-E^2)/g_0}/R$ [16], comparison with Eq. (3.18) provides a relation to have identical configurations at intermediate values

$$1-\left(\frac{E}{g_0}\right)^2 = \frac{\left[1-(G_0\theta_0)^2\right]^2 - \left(\frac{\hat{E}}{g_0}\right)^2}{\left[1-(G_0\theta_0)^2\right]\left[1+\frac{G_0\theta_0\frac{\hat{E}}{g_0}}{1-(G_0\theta_0)^2}\right]^2}. \tag{3.20}$$

We examine the equation (3.16) by dividing the cases into three, i.e., $\hat{\omega}^2 > 0$, $\hat{\omega}^2 = 0$, and $\hat{\omega}^2 < 0$. As a boundary condition of $\hat{\tau}(x)$, we use $\hat{\tau}(0) = 0$ without losing generality.

3.1 Array of kink and antikink for $\hat{\omega}^2 > 0$

When $\hat{\omega}^2 > 0$, nontrivial solution can exist for positive $\mathcal{E}_1$ which leads to $\hat{\mathcal{T}}_1^2/(G_0^2 - \hat{E}^2)(-\hat{T}^{11})^2 > 1$, and it is an oscillating configuration for any runaway tachyon potential $\hat{V}$ with $\hat{V}(\hat{\tau} = 0) = 1$ and $\hat{V}(\hat{\tau} = \pm\infty) = 0$. Under the specific potential (2.20), we obtain an exact solution

$$\frac{\hat{\tau}(x)}{R} = \pm\sqrt{\frac{\hat{\mathcal{T}}_1^2}{(G_0^2 - \hat{E}^2)(-\hat{T}^{11})^2} - 1}\, \sin(\hat{\omega}x). \tag{3.21}$$

The NC tachyon field oscillates sinusoidally between maximum and minimum values, $\pm R\sqrt{\frac{\hat{\mathcal{T}}_1^2}{(G_0^2-\hat{E}^2)(-\hat{T}^{11})^2} - 1}$ with period $2\pi/\hat{\omega} = 2\pi R\sqrt{G_0(1+\hat{E}\theta_0)^2/(G_0^2 - \hat{E}^2)}$. Then this solution is interpreted as the array of kink and antikink in the presence of NC electric field $\hat{E}$ transverse to the kink (or antikink). Note that the period does not depend on the an integration constant $-\hat{T}^{11}$. This property is unique under the specific form of tachyon potential (3.8). A proof is simple as given in the following. Once we identify the system described by Eq. (3.16) as that of a hypothetical particle with unit mass, of which position is $\hat{\tau}$ and time x, then possible motion is nothing but that of a one-dimensional simple harmonic oscillator and its period $2\pi/\hat{\omega}$ is independent of mechanical energy $\mathcal{E}_1$. Such $\mathcal{E}_1$-independency holds uniquely for the simple harmonic oscillator as proved in Ref. [37].

This phenomenon can also be seen in the NC action (3.7) through a rescaling of spatial coordinate $x \to \chi = \sqrt{(G_0^2 - \hat{E}^2)/G_0(1+\hat{E}\theta_0)^2}\, x$

$$\hat{S} = -\sqrt{G_0}\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int dtdx\,\hat{V}\sqrt{\frac{G_0^2 - \hat{E}^2}{G_0(1+\hat{E}\theta_0)^2} + \hat{V}^2\hat{\tau}'^2} \tag{3.22}$$

$$= -\sqrt{G_0}\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int dtd\chi\,\hat{V}\sqrt{1 + \hat{V}^2\left(\frac{d\hat{\tau}}{d\chi}\right)^2}. \tag{3.23}$$

Formal resemblance between Eq. (3.23) and the rescaled action in ordinary effective action with constant electric field is clear under the point transformation (2.16) and the relation (3.6) [16]. In the pure NCFT limit with vanishing $\hat{E}$, an exact identification is made by the relation between $\mathcal{T}_1$ and $\hat{\mathcal{T}}_1$ (3.4).

Substituting the equation of motion (3.16) and kink solution (3.21) into the NC action (3.22) for half period, we obtain the formula for tension of a unit kink (or unit antikink)

$$\frac{\hat{S}}{-\int dt\sqrt{G_0}} = \frac{\hat{\mathcal{T}}_1^2}{-\hat{T}^{11}\sqrt{G_0}}\int_{-\frac{\pi}{2\hat{\omega}}}^{\frac{\pi}{2\hat{\omega}}} dx\,\hat{V}^2 \tag{3.24}$$

$$
\begin{aligned}
&= \pi R\hat{\mathcal{T}}_1(1+\hat{E}\theta_0) && (3.25)\\
&\equiv \hat{\mathcal{T}}_0(1+\hat{E}\theta_0), && (3.26)
\end{aligned}
$$

where a relation from the open string metric (3.3),

$$
\int dtdx\sqrt{-\det(G_{\mu\nu})} = \int dtdxG_0 = \int dt\sqrt{G_0}\times\int dx\sqrt{G_0}, \tag{3.27}
$$

was used in the left-hand side of Eq. (3.24) and will be used in the formulas of Hamiltonian and tension. The tension is corrected by a factor $(1+\hat{E}\theta_0)$ when the electric field $\hat{E}$ exists in NC spacetime, however it reduces to the value $\hat{\mathcal{T}}_0$ in the limit of either vanishing electric field ($\hat{E}\to 0$) or pure commutative spacetime ($\theta_0\to 0$). The obtained decent relation in NCFT (3.25)–(3.26) is identical to $\mathcal{T}_0 = \pi R\mathcal{T}_1$ in ordinary EFT irrespective of the value of $\hat{E}$ and θ_0. The Hamiltonian for a single kink (or antikink) in the array is

$$
\begin{aligned}
H &\equiv \int_{-\frac{\pi}{2\hat{\omega}}}^{\frac{\pi}{2\hat{\omega}}} dx\sqrt{G_0}\,(-\hat{T}^0_{\ 0}) && (3.28)\\
&= \frac{\pi}{\hat{\omega}}\sqrt{G_0}\,\hat{\Pi}\frac{\hat{E}}{G_0} + \pi R\hat{\mathcal{T}}_1(1+\hat{E}\theta_0). && (3.29)
\end{aligned}
$$

Comparing it to the tension from the action (3.26), the second term in Eq. (3.29) is the tension of a single D0-brane (or $\bar{\mathrm{D}}_0$-brane) and the first term comes from F1 string fluid of which the signal appears through nonvanishing NC electric field $\hat{E}$.

Substituting the array solution (3.21) into the energy density (3.15), we have

$$
-\hat{T}^0_{\ 0} = \hat{\Pi}\frac{\hat{E}}{G_0} + \frac{\hat{E}/G_0}{\hat{\Pi}}\frac{\hat{\mathcal{T}}_1^2}{1+\left[\frac{\hat{\mathcal{T}}_1^2}{\hat{\Pi}^2\left[\left(\frac{G_0}{\hat{E}}\right)^2-1\right]}-1\right]\sin^2\left[\sqrt{\frac{G_0^2-\hat{E}^2}{G_0(1+\hat{E}\theta_0)^2}}\,\frac{x}{R}\right]} \tag{3.30}
$$

composed of a constant density from the first string fluid term and an oscillating contribution from the second array term, having values between $\hat{\Pi}\left[1-(\hat{E}/G_0)^2\right]/(\hat{E}/G_0)$ and $\hat{\mathcal{T}}_1^2(\hat{E}/G_0)/\hat{\Pi}$.

From now on, let us take various limits. (i) Taking the NC parameter to be zero ($\theta_0\to 0$) keeping $\hat{E}$ and $\hat{\Pi}$ fixed, we obtain the limit of EFT with $G_0 = g_0$ smoothly [16, 18]. (ii) Turning off the NC electric field ($\hat{E}\to 0$) keeping G_0, θ_0, and $-\hat{T}^{11}$ fixed, we easily confirm disappearance of F1 contribution in pure NCFT limit

$$
-\hat{T}^0_{\ 0}\Big|_{\hat{E}\to 0} = \hat{\mathcal{T}}_1\frac{\left(\frac{\hat{\mathcal{T}}_1}{-\hat{T}^{11}G_0}\right)}{1+\left[\left(\frac{\hat{\mathcal{T}}_1}{-\hat{T}^{11}G_0}\right)^2-1\right]\sin^2\left(\sqrt{G_0}\frac{x}{R}\right)}. \tag{3.31}
$$

It is consistent with vanishing F1 charge density in this limit, i.e., $\lim_{\hat{E}\to 0}\hat{\Pi} = \lim_{\hat{E}\to 0}(-\hat{T}^{11})\hat{E} \overset{\text{finite }(-\hat{T}^{11})}{=} 0$. (iii) The limit of zero thickness is achieved by taking $\hat{\Pi}\to 0$ with fixed G_0 and $\hat{E}$, where the energy density (3.30) from the NC action (2.17) is given by a sum of δ-functions

$$-\hat{T}^0_{\ 0}\Big|_{\hat{\Pi}\to 0} = \hat{\mathcal{T}}_0(1+\hat{E}\theta_0)\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{G_0}}\delta\left(x-\frac{n\pi}{\hat{\omega}}\right). \tag{3.32}$$

Since $\hat{\Pi}\to 0$ limit with fixed $\hat{E}$ is equivalent to vanishing pressure limit $-\hat{T}^{11}\to 0$ and each δ-function in Eq. (3.32) stands for the energy density of each kink or antikink, this zero thickness limit corresponds to BPS limit in NCFT, which was the case in ordinary EFT [14, 16]. From Eq. (3.21), each kink becomes a topological kink connecting two true vacua at $\hat{\tau}=\pm\infty$ though the configuration is singular. Note that we have used $\hat{V}(0)=1$ and $\hat{V}(\pm\infty)=0$ to get the BPS limit, which implies that it is saturated for any runaway potential [14].

3.2 Thick topological BPS kink for $\hat{\omega}^2 = 0$

When $\hat{\omega}^2=0$, a drastic change is made due to disappearance of the potential term $U_1=0$ in Eq. (3.16), and then we find a linear solution

$$\hat{\tau}(x) = \pm\frac{\hat{\mathcal{T}}_1}{\sqrt{G_0}(-\hat{T}^{11})(1+\hat{E}\theta_0)}\,x, \tag{3.33}$$

where $\hat{\tau}(0)=0$. Therefore, the obtained solution can also be understood as the infinite period limit of the periodic solution (3.21) and then it describes single topological kink (or antikink) connecting two vacua at $\hat{\tau}=\infty$ and $\hat{\tau}=-\infty$ smoothly. Note that, for finite E_0 and $\hat{E}$, formula of $\hat{\omega}$ (3.18) leads to

$$G_0^2 = \hat{E}^2 \tag{3.34}$$

so that the solution (3.33) is rewritten by

$$\hat{\tau}(x) = \pm\frac{\hat{\mathcal{T}}_1\sqrt{\hat{E}}}{\hat{\Pi}(1+\hat{E}\theta_0)}\,x. \tag{3.35}$$

Let us read the tension of this kink or antikink from the action (3.22) as follows:

$$\frac{\hat{S}}{-\int dt\sqrt{G_0}} = \hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int_{-\infty}^{\infty}dx\,\hat{V}(\hat{\tau})\sqrt{\hat{\omega}^2R^2+\hat{V}^2\hat{\tau}'^2} \tag{3.36}$$

$$\overset{\hat{\omega}^2=0}{=} \hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int_{-\infty}^{\infty}d\hat{\tau}\,\hat{V}(\hat{\tau})^2 \tag{3.37}$$

$$\overset{\hat{\tau}/R=\sinh(\hat{T}/R)}{=} \quad \hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int_{-\infty}^{\infty} d\hat{T}\, V(\hat{T}) \tag{3.38}$$

$$= \quad \hat{\mathcal{T}}_0(1+\hat{E}\theta_0), \tag{3.39}$$

where we used Eq. (3.8) for $\hat{V}(\hat{\tau})$ in the second line and Eq. (2.15) for $V(\hat{T})$ in the third line. The expression (3.38) coincides with the exact integral formula of the tension for both the singular BPS kink [14] and the thick BPS kink with critical electric field [16, 18, 38].

For the topological kink, the pressure $\hat{T}^{11}$ is provided by a constant background of the F1, $-\hat{T}^{11} = \hat{\Pi}/G_0$, and the Hamiltonian saturates the BPS relation, which is expressed by the sum of the F1 charge and the D0 tension:

$$H \equiv \int_{-\infty}^{\infty} dx \sqrt{G_0}\,(-\hat{T}^0_{\ 0}) \tag{3.40}$$

$$= \int_{-\infty}^{\infty} dx \sqrt{G_0}\,\hat{\Pi} + \hat{\mathcal{T}}_0(1+\hat{E}\theta_0). \tag{3.41}$$

The energy density of D0 is localized near $x = 0$ as

$$-\hat{T}^0_{\ 0} = \hat{\Pi} + \frac{1}{\sqrt{G_0}}\hat{\mathcal{T}}_0(1+\hat{E}\theta_0)\frac{\xi/\pi}{x^2+\xi^2}, \tag{3.42}$$

where the width ξ is

$$\xi = \frac{\hat{\Pi}}{\hat{E}}\frac{\sqrt{\hat{E}}R(1+\hat{E}\theta_0)}{\hat{\mathcal{T}}_1}. \tag{3.43}$$

(i) The limit of ordinary EFT is achieved by making θ_0 or equivalently E_0 vanish with fixed $\hat{E}$ and $\hat{\Pi}$. (ii) If we take the limit of pure NCFT ($\hat{E} \to 0$), the open string metric becomes singular, $G_0 \to 0$ with keeping the closed string variables finite, $G_0\theta_0 = \hat{E}\theta_0 = E_0/g_0 = 1$. Then the pressure diverges, $-\hat{T}^{11} \propto 1/\sqrt{G_0} \to \infty$, and the F1 charge density vanishes, $\hat{\Pi} = -\hat{T}^{11}G_0 \to 0$. On the other hand, the tachyon configuration is smooth with finite slope

$$\hat{\tau}(x) = \pm\frac{\hat{\mathcal{T}}_1}{2\sqrt{G_0}(-\hat{T}^{11})}x, \tag{3.44}$$

and its tension $\hat{\mathcal{T}}_0(1+\hat{E}\theta_0)$ is also finite when $\hat{\mathcal{T}}_1$ is finite

$$\hat{\mathcal{T}}_0(1+\hat{E}\theta_0) = \pi R\hat{\mathcal{T}}_1(1+\hat{E}\theta_0) = 2\pi R\hat{\mathcal{T}}_1. \tag{3.45}$$

This result is noteworthy because the codimension-one D-brane is given by a thick smooth topological BPS object with finite tension despite an infinite NC parameter, $\theta_0 \to \infty$, and singular open string metric, $G_0 \to 0$. In terms of closed string variables g_0 and E_0, this limit can be understood as that of critical electric field $E_0 = g_0$ for nonvanishing g_0.

Finiteness of the tension $2\hat{\mathcal{T}}_0$ means vanishing tension $\mathcal{T}_0$ in the EFT due to Eq. (3.38) and the relation (3.4). (iii) For finite G_0 or equivalently for finite $\hat{E}$, the F1 charge density $\hat{\Pi}$ governs the width of kink ξ: As $\hat{\Pi}$ goes to zero the localized piece of the energy density (3.42) approaches a δ-function for finite tension, while large $\hat{\Pi}$ broadens the width.

3.3 Topological kink for $\hat{\omega}^2 < 0$ ($\hat{E} > G_0 > 0$)

When $\hat{\omega}^2 < 0$, the potential $U_1(\hat{\tau})$ in Eq. (3.18) becomes upside down leading to a hyperbolic solution. Since the metric component G_0 in Eq. (3.4) is nonnegative, we examine Eq. (3.16) for the range $\hat{E} > G_0 > 0$. For this range, the solution of the equation (3.16) is

$$\frac{\hat{\tau}(x)}{R} = \pm\sqrt{\frac{\hat{\mathcal{T}}_1^2}{(\hat{E}^2 - G_0^2)(-\hat{T}^{11})^2} + 1}\,\sinh(|\hat{\omega}|x). \tag{3.46}$$

The reason why we obtain a hyperbolic configuration is clearly seen by looking at the action (3.7) with a rescaling of x

$$\hat{S} = -\int dt\sqrt{G_0}\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int_{-\infty}^{\infty} dx\,\hat{V}(\hat{\tau})\sqrt{\hat{V}^2\hat{\tau}'^2 - |\hat{\omega}^2|R^2} \tag{3.47}$$

$$= -\int dt\sqrt{G_0}\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int_{-\infty}^{\infty} d\eta\,\hat{V}\sqrt{\hat{V}(\hat{\tau})^2\left(\frac{d\hat{\tau}}{d\eta}\right)^2 - 1}$$

$$\overset{\hat{\tau}/R=\sinh(\hat{T}/R)}{=} -\int dt\sqrt{G_0}\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\int_{-\infty}^{\infty} d\eta\,V(\hat{T})\sqrt{\left(\frac{d\hat{T}}{d\eta}\right)^2 - 1}\,, \tag{3.48}$$

where

$$\eta = |\hat{\omega}|Rx = \sqrt{\frac{\hat{E}^2 - G_0^2}{G_0(1+\hat{E}\theta_0)^2}}\,x. \tag{3.49}$$

The rescaled action (3.48) resembles formally that for a time-dependent rolling tachyon solution [18, 23] once we identify η as time and $\hat{T}$ as T in EFT. Since $(d\hat{T}/d\eta)^2 - 1$ occurs in the square root instead of $1-(d\hat{T}/d\eta)^2$, hyperbolic sine solution (3.46) is only allowed.

Again the tension is computed from the action by substituting the solution (3.46)

$$\frac{\hat{S}}{-\int dt\sqrt{G_0}} = \frac{\hat{E}\hat{\mathcal{T}}_1^2}{\hat{\Pi}G_0}\int_{-\infty}^{\infty} dx\sqrt{G_0}\,\hat{V}^2 \tag{3.50}$$

$$= 2R\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)\arctan\left(\frac{\hat{E}\hat{\mathcal{T}}_1}{\hat{\Pi}\sqrt{\hat{E}^2 - G_0^2}}\right), \tag{3.51}$$

which is less than $\pi R\hat{\mathcal{T}}_1(1+\hat{E}\theta_0)$ and approaches this maximum value as $\hat{E}^2 \to G_0^2$ or $\hat{\Pi} \to 0$. Note that $\sqrt{-X}$ from Eq. (3.9) or equivalently the square root in Eq. (3.48) is

always kept to be real and finite for making the kink solution (3.46) and the action (3.47) meaningful. The energy density (3.15) is divided by a constant part from F1 fluid and a localized piece from the $\hat{V}^2$ term

$$-\hat{T}^0_{\ 0} = \hat{\Pi}\frac{\hat{E}}{G_0} + \frac{\hat{E}/G_0}{\hat{\Pi}} \frac{\hat{\mathcal{T}}_1^2}{1 + \left[\frac{\hat{\mathcal{T}}_1^2}{\hat{\Pi}^2\left[1-\left(\frac{G_0}{\hat{E}}\right)^2\right]} + 1\right]\sinh^2\left[\sqrt{\frac{\hat{E}^2-G_0^2}{G_0(1+\hat{E}\theta_0)^2}}\,\frac{x}{R}\right]}. \tag{3.52}$$

Despite the infinite slope of the tachyon profile $\hat{\tau}(x)$ (3.46) at $x = \pm\infty$, the localized D$(p-1)$ part of energy density (3.52) decreases exponentially to zero at the asymptotic regions. The tension computed from the localized piece of $-\hat{T}^0_{\ 0}$ (3.52) coincides exactly with the value in Eq. (3.51). Though its existence seems unconventional due to the value of electric field $\hat{E}$ larger than the critical value, the obtained kink has correspondence with a topological kink in Ref. [16].

Let us discuss various limits in what follows. (i) The limit of EFT is smoothly taken by $\theta_0 \to 0$ with fixed G_0, which corresponds to $E_0 \to 0$ with fixed $g_0 = G_0$. (ii) Pure NCFT limit is achieved in vanishing interpolating field limit $\hat{E} \to 0$. Then $\hat{E} > G_0 > 0$ condition leads to $G_0 \to 0$. Since consistency asks $\hat{E}^2/G_0 \to 0$, $\hat{\omega}^2 = (G_0 - \hat{E}^2/G_0)/R^2 \to 0^-$. Therefore, it reduces to the same $\hat{E} \to 0$ limit of the previous subsection 3.2, and the object is nothing but the thick topological BPS kink in pure NCFT with $\theta_0 \to \infty$ and $\hat{E}\theta_0 = 1$. (iii) Finally we take thin limit by taking $\hat{\Pi} \to 0$ for $\hat{E} > 0$. Then we have $-\hat{T}^{11} = \hat{\Pi}/\hat{E} \to 0$ which means the coefficient in front of hyperbolic sine function in Eq. (3.46) is singular but the slope inside it remains finite. In the expression of energy density (3.52), constant contribution of the F1 vanishes and a sharply peaked localized piece cannot become a δ-function, while its tension recovers $\hat{\tau}_0 = \pi R\hat{\tau}_1(1 + \hat{E}\theta_0)$.

We have obtained, from an unstable D1-brane, all possible static codimension-one solitons identified as D0-branes. Let us finish this section by summarizing the obtained kinks in a table.

range of parameters	soliton species	type of solution
$\hat{\omega}^2 > 0$	array of kink-antikink	sinusoidal
$\hat{\omega}^2 = 0$	topological kink (BPS)	linear
$\hat{\omega}^2 < 0$: $\hat{E} > G_0 > 0$	topological kink with $\hat{\tau}(\pm\infty) = \pm\infty$	hyperbolic sine

Table 1: List of static solitons of codimension-one.

4 Conclusion and Discussion

In this paper we considered a real NC tachyon coupled to an NC U(1) gauge field describing NCFT version of the dynamics of an unstable Dp-brane, and proposed its action as DBI type which is different from quadratic one already studied. For slowly varying NC gauge field and NC tachyon, we showed up to the leading NC parameter, the equivalence between the proposed NC tachyon action and the DBI type effective action of ordinary tachyon field.

For a flat unstable D1-brane with arbitrary diagonal component of open string metric G_0, NC parameter θ_0, and interpolating electric field $\hat{E}$, we found all the static kinks solutions, i.e., they are array of kink-antikink and single topological kink. The existence of kink solutions is universal since they are supported irrespective of any ambiguity in assigning star products between the NC fields, symmetrization procedure among the star-producted terms, and detailed shape of runaway NC tachyon potential. For a specific tachyon potential, the kinks are given as exact solutions of which functional forms coincide exactly with BCFT tachyon profiles. Computing the tension of unit kink (or unit antikink), the obtained kinks are identified with array of D0$\bar{\text{D}}$0 or single D0.

When $G_0^2 = \hat{E}^2$, there exists single topological kink saturating BPS bound despite its nonzero thickness. In the limit of singular open string metric $G_0 \to 0$ or that of divergent NC parameter $\theta_0 \to \infty$ due to $G_0\theta_0 = 1$, finiteness of the tension of NC kink requires vanishing tension of kink in ordinary EFT.

The present study of NC kinks demonstrate clearly a relation for D0 and their composites from an unstable D1 among NCFT, ordinary EFT [15, 16, 17, 18], and BCFT [22]. Therefore, further study on NC tachyon is needed in terms of other languages for off-shell string theory calculation, *e.g.*, string field theories [39].

Though our discussion was restricted to the case of D1-brane, for pure electric case with parallel $\hat{\mathbf{E}}$ and NC parameter $\vec{\theta}$, an extension to Dp-brane of arbitrary p is straightforward by choosing the transverse direction to D$(p-1)$-brane as that of $\hat{\mathbf{E}}$. For the general flat unstable Dp-branes with various NC parameters including spatial θ^{ij}'s, the analysis becomes complicated. It appears therefore that the D2 example might provide some useful hints towards the solution of the more general problem.

As we made a smooth bridge from the kinks in ordinary EFT to those in pure NCFT by employing the interpolating electric field $\hat{\mathbf{E}}$, the same relation can be constructed for other tachyon solitons like thick tachyon tubes [32] or homogeneous time dependent solutions represented by rolling tachyons [23]. For spike (or BIon) configurations, thin NC solutions are known [40] but any thick tachyon spike solutions are absent both in EFT

and NCFT up to now.

Acknowledgements

The authors would like to thank J.L. Karczmarek, Chanju Kim, Chong Oh Lee, Sangmin Lee, Y. Michishita, and A. Sen for valuable discussions. This work was supported by funds from Sungkyunkwan University (R.B.) and is the result of research activities (Astrophysical Research Center for the Structure and Evolution of the Cosmos (ARCSEC)) supported by Korea Science & Engineering Foundation (Y.K. and O.K.).

A Equivalence between DBI action with tachyon and NCDBI action with NC tachyon

In this appendix we prove the equivalence between the DBI-type tachyon effective action and its NC version given in Eq. (2.17) up to leading order in NC parameter θ. The proof is similar in spirit to that carried out in [2] for the non-tachyonic models.

DBI type tachyon effective action in the presence of the NS-NS field on an unstable Dp-brane is given in Eq. (2.13). For slowly varying NC fields, $\hat{F}_{\mu\nu}$ and $\hat{D}_\mu\hat{T}$, the NC counterpart of the action (2.13) is proposed in Eq. (2.17). Here we consider the general case including an interpolating field $\Phi_{\mu\nu}$, analogous to the case without the NC tachyon (2.6);

$$\hat{S} = -\frac{1}{G_s(2\pi)^{\frac{p-1}{2}}}\int d^{p+1}x\,\hat{V}(\hat{T})\sqrt{-\hat{X}_\Phi}, \tag{A.1}$$

where

$$\hat{X}_\Phi = \det\left(G_{\mu\nu} + \Phi_{\mu\nu} + \hat{F}_{\mu\nu} + \hat{D}_\mu\hat{T}\hat{D}_\nu\hat{T}\right). \tag{A.2}$$

The variables in the two actions (2.13) and (A.1) are related by

$$\frac{1}{G+\Phi} + \theta = \frac{1}{g+B}, \tag{A.3}$$

$$G_s = g_s\sqrt{\frac{-\det(G+\Phi)}{-\det(g+B)}}. \tag{A.4}$$

Using the SW map for the gauge field $\hat{A}$ [2] and its extension to NC tachyon field $\hat{T}$, the two actions (2.13) and (A.1) will be shown to satisfy the following relation

$$\mathcal{L} = \hat{\mathcal{L}} + \text{total derivative}. \tag{A.5}$$

In order to prove the relation (A.5), we perform a transformation of NC parameter, $\theta \to \theta + \delta\theta$, for fixed g_s, $g_{\mu\nu}$ and $B_{\mu\nu}$, and see the small variation of the action (A.1). The SW maps for the gauge and real scalar fields are given by [2, 41, 42]

$$\begin{aligned}
\hat{A}_\mu &= A_\mu - \frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa,\ \partial_\lambda A_\mu + F_{\lambda\mu}\} + \mathcal{O}(\theta^2), && \text{(A.6)}\\
\hat{F}_{\mu\nu} &= F_{\mu\nu} + \frac{1}{4}\theta^{\kappa\lambda}\left(2\{F_{\mu\kappa},\ F_{\nu\lambda}\} - \{A_\kappa,\ D_\lambda F_{\mu\nu} + \partial_\lambda F_{\mu\nu}\}\right) + \mathcal{O}(\theta^2), && \text{(A.7)}\\
\hat{T} &= T - A_\alpha\theta^{\alpha\beta}\partial_\beta T + \mathcal{O}(\theta^2), && \text{(A.8)}\\
\hat{D}_\mu\hat{T} &= \partial_\mu T - F_{\mu\alpha}\theta^{\alpha\beta}\partial_\beta T - A_\alpha\theta^{\alpha\beta}\partial_\mu\partial_\beta T + \mathcal{O}(\theta^2). && \text{(A.9)}
\end{aligned}$$

Variation of Lagrange density $\hat{\mathcal{L}}$ in Eq. (A.1) with respect to θ is given by

$$\delta\hat{\mathcal{L}} = \left[-\frac{\delta G_s}{G_s} + \frac{1}{\hat{V}}\delta\hat{V} + \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\hat{X}}\left(\delta G + \delta\Phi + \delta\hat{F} + \delta(\hat{D}\hat{T}\hat{D}\hat{T})\right)\right)\right]\hat{\mathcal{L}}, \quad \text{(A.10)}$$

where we used matrix notation such as $(AB)_{\mu\nu} = A_{\mu\lambda}B^\lambda{}_\nu$ and $\mathrm{Tr}(AB) = A_{\mu\lambda}B^{\lambda\mu}$. From Eqs. (A.3)–(A.4), we have

$$\begin{aligned}
\delta G_s &= \frac{1}{2}g_s\sqrt{\frac{-\det(G+\Phi)}{-\det(g+B)}}\,\mathrm{Tr}\left(\delta\theta(G+\Phi)\right) = \frac{1}{2}G_s\mathrm{Tr}\left(\delta\theta(G+\Phi)\right), && \text{(A.11)}\\
\delta(G+\Phi) &= (G+\Phi)\delta\theta(G+\Phi). && \text{(A.12)}
\end{aligned}$$

The SW maps for the NC fields in Eqs. (A.6)–(A.9) lead to

$$\begin{aligned}
\delta\hat{F}_{\mu\nu} &= -\delta\theta^{\alpha\beta}(\hat{F}_{\mu\alpha} * \hat{F}_{\beta\nu}) - \frac{1}{2}\delta\theta^{\alpha\beta}\hat{A}_\alpha * (\hat{D}_\beta\hat{F}_{\mu\nu} + \partial_\beta\hat{F}_{\mu\nu}), && \text{(A.13)}\\
\delta\hat{T} &= -\frac{1}{2}\delta\theta^{\alpha\beta}(\hat{A}_\alpha * \partial_\beta\hat{T} + \partial_\beta\hat{T} * \hat{A}_\alpha), && \text{(A.14)}\\
\delta\hat{D}_\mu\hat{T} &= -\frac{1}{2}\delta\theta^{\alpha\beta}\left(\hat{F}_{\mu\alpha} * \partial_\beta\hat{T} + \partial_\beta\hat{T} * \hat{F}_{\mu\alpha} + \hat{A}_\alpha * \partial_\mu\partial_\beta\hat{T} + \partial_\mu\partial_\beta\hat{T} * \hat{A}_\alpha\right), && \text{(A.15)}
\end{aligned}$$

where we used the following property of star product

$$\delta\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}(\hat{f} * \hat{g}) = \frac{1}{2}\delta^{\mu\nu}\partial_\mu\hat{f} * \partial_\nu\hat{g}. \quad \text{(A.16)}$$

Since proving the relation (A.5) up to the leading order in θ is of our interest, insertion of Eqs. (A.6)–(A.9) into Eqs. (A.13)–(A.15) results in

$$\begin{aligned}
\delta\hat{F}_{\mu\nu} &= -(F\delta\theta F)_{\mu\nu} - A_\alpha\delta\theta^{\alpha\beta}\partial_\beta F_{\mu\nu} + \mathcal{O}(\theta^2), && \text{(A.17)}\\
\delta\hat{T} &= -A_\alpha\delta\theta^{\alpha\beta}\partial_\beta T + \mathcal{O}(\theta^2), && \text{(A.18)}\\
\delta\hat{D}_\mu\hat{T} &= -F_{\mu\alpha}\delta\theta^{\alpha\beta}\partial_\beta T - A_\alpha\delta\theta^{\alpha\beta}\partial_\mu\partial_\beta T + \mathcal{O}(\theta^2). && \text{(A.19)}
\end{aligned}$$

Substituting Eqs. (A.17)–(A.19) into Eq. (A.10), we get

$$
\begin{aligned}
\delta\hat{\mathcal{L}} &= \Bigg[\frac{1}{\hat{V}}\frac{d\hat{V}}{d\hat{T}}\delta\hat{T} - \frac{1}{2}\mathrm{Tr}\left(\delta(G+\Phi)\right) + \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\hat{X}}(G+\Phi)\delta\theta(G+\Phi)\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\hat{X}}\delta\hat{F}\right) + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}\left(\delta(\hat{D}_\mu\hat{T})\hat{D}_\nu\hat{T} + \hat{D}_\mu\hat{T}\delta(\hat{D}_\nu\hat{T})\right)\Bigg]\hat{\mathcal{L}} \\
&= -\Bigg[\frac{1}{\hat{V}}\frac{d\hat{V}}{d\hat{T}}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta T + \frac{1}{2}\mathrm{Tr}(\hat{F}\delta\theta) - \frac{1}{2}\mathrm{Tr}\left(\frac{1}{\hat{X}}\hat{F}\delta\theta\hat{D}\hat{T}\hat{D}\hat{T} + \frac{1}{\hat{X}}\hat{D}\hat{T}\hat{D}\hat{T}\delta\theta\hat{F}\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta F_{\mu\nu} + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}\left((F\delta\theta\partial T)_\mu\partial_\nu T + \partial_\mu T(F\delta\theta\partial T)_\nu\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}\left(A_\alpha\delta\theta^{\alpha\beta}\partial_\beta\partial_\mu T\partial_\nu T + A_\alpha\delta\theta^{\alpha\beta}\partial_\beta\partial_\nu T\partial_\mu T\right)\Bigg]\hat{\mathcal{L}}, \qquad \text{(A.20)}
\end{aligned}
$$

where in the second line we used the following relations

$$
\begin{aligned}
&\mathrm{Tr}\left(\frac{1}{\hat{X}}(G+\Phi)\delta\theta(G+\Phi) - \delta\theta(G+\Phi)\right) + \mathrm{Tr}\left(\frac{1}{\hat{X}}\delta\hat{F}\right) \\
&\quad = -\mathrm{Tr}\left(\frac{1}{\hat{X}}(\hat{F}+\hat{D}\hat{T}\hat{D}\hat{T})\delta\theta(\hat{X}-\hat{F}-\hat{D}\hat{T}\hat{D}\hat{T})\right) - \mathrm{Tr}\left(\frac{1}{\hat{X}}\hat{F}\delta\theta\hat{F}\right) \\
&\qquad - \left(\frac{1}{\hat{X}}\right)^{\nu\mu}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta F_{\mu\nu} \\
&\quad = -\mathrm{Tr}(\hat{F}\delta\theta) + \mathrm{Tr}\left(\frac{1}{\hat{X}}\hat{F}\delta\theta\hat{D}\hat{T}\hat{D}\hat{T} + \frac{1}{\hat{X}}\hat{D}\hat{T}\hat{D}\hat{T}\delta\theta\hat{F}\right) - \left(\frac{1}{\hat{X}}\right)^{\nu\mu}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta F_{\mu\nu},
\end{aligned}
$$

and,

$$
\mathrm{Tr}\left(\frac{1}{\hat{X}}\hat{F}\hat{D}\hat{T}\hat{D}\hat{T}\delta\theta\hat{D}\hat{T}\hat{D}\hat{T}\right) = 0.
$$

Up to the leading order in θ, we obtain

$$
\begin{aligned}
&\mathrm{Tr}\left(\frac{1}{\hat{X}}\hat{F}\delta\theta\hat{D}\hat{T}\hat{D}\hat{T} + \frac{1}{\hat{X}}\hat{D}\hat{T}\hat{D}\hat{T}\delta\theta\hat{F}\right) \\
&\quad = \left(\frac{1}{\hat{X}}\right)^{\nu\mu}\frac{(F\delta\theta\partial T)_\mu\partial_\nu T + \partial_\mu T(F\delta\theta\partial T)_\nu}{1+(\hat{T}/R)^2} + \mathcal{O}(\theta^2), \\
&A_\alpha\delta\theta^{\alpha\beta}\partial_\beta\hat{\mathcal{L}} \\
&\quad = \Bigg[\left(\frac{1}{\hat{V}}\frac{d\hat{V}}{d\hat{T}}\right)A_\alpha\delta\theta^{\alpha\beta}\partial_\beta\hat{T} + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta\hat{F}_{\mu\nu} \\
&\qquad + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}\left(A_\alpha\delta\theta^{\alpha\beta}\partial_\beta(\hat{D}_\mu\hat{T})\hat{D}_\nu\hat{T} + \hat{D}_\mu\hat{T}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta(\hat{D}_\nu\hat{T})\right)\Bigg]\hat{\mathcal{L}}. \\
&\quad = \Bigg[\left(\frac{1}{\hat{V}}\frac{d\hat{V}}{d\hat{T}}\right)A_\alpha\delta\theta^{\alpha\beta}\partial_\beta T + \frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}A_\alpha\delta\theta^{\alpha\beta}\partial_\beta F_{\mu\nu}
\end{aligned}
$$

$$+\frac{1}{2}\left(\frac{1}{\hat{X}}\right)^{\nu\mu}\left(A_\alpha\delta\theta^{\alpha\beta}\partial_\beta(\partial_\mu T)\partial_\nu T+\partial_\mu T A_\alpha\delta\theta^{\alpha\beta}\partial_\beta(\partial_\nu T)\right)\Big]\hat{\mathcal{L}}+\mathcal{O}(\theta^2). \quad (A.21)$$

Then, with the help of Eqs. (A.20)–(A.21), the variation of Lagrange density (A.10) is expressed by total derivative terms for leading order in θ;

$$\begin{aligned}\delta\hat{\mathcal{L}} &= -\frac{1}{2}\hat{\mathcal{L}}F_{\alpha\beta}\delta\theta^{\beta\alpha}-A_\alpha\delta\theta^{\alpha\beta}\partial_\beta\hat{\mathcal{L}}+\mathcal{O}(\theta^2)\\ &= -\partial_\beta(A_\alpha\delta\theta^{\alpha\beta}\hat{\mathcal{L}})+\mathcal{O}(\theta^2). \end{aligned} \quad (A.22)$$

This completes the proof of the relation (A.5) up to leading order in θ.

B Derivation of NC Equations of Motion

B.1 NC tachyon equation

Let us consider variation of the NC action (2.17) when a small variation of the NC tachyon field $\hat{T}$ is taken;

$$\hat{S}[\hat{T}+\delta\hat{T}]-\hat{S}[\hat{T}]=\int d^{p+1}x\,\frac{\delta\hat{S}[\hat{T}]}{\delta\hat{T}(x)}\delta\hat{T}(x)=\delta_{\hat{T}}\hat{S}_1[\hat{T}]+\delta_{\hat{T}}\hat{S}_2[\hat{T}], \quad (A.23)$$

where

$$\delta_{\hat{T}}\hat{S}_1[\hat{T}] = -\hat{\mathcal{T}}_p\int d^{p+1}x\,\delta_{\hat{T}}\hat{V}(\hat{T})\sqrt{-\hat{X}}, \quad (A.24)$$

$$\delta_{\hat{T}}\hat{S}_2[\hat{T}] = -\hat{\mathcal{T}}_p\int d^{p+1}x\,\hat{V}(\hat{T})\,\delta_{\hat{T}}\sqrt{-\hat{X}}. \quad (A.25)$$

From the definition of the NC tachyon potential (2.20) we obtain

$$\begin{aligned}\delta_{\hat{T}}\hat{V}(\hat{T}) &= \hat{V}(\hat{T}+\delta\hat{T})-\hat{V}(\hat{T})\\ &= -\frac{1}{2}\left(\frac{\delta\hat{T}}{R}*\frac{\hat{T}}{R}+\frac{\hat{T}}{R}*\frac{\delta\hat{T}}{R}\right)\\ &\quad +\frac{5}{24}\left(\frac{\delta\hat{T}}{R}*\frac{\hat{T}}{R}*\frac{\hat{T}}{R}*\frac{\hat{T}}{R}+\frac{\hat{T}}{R}*\frac{\delta\hat{T}}{R}*\frac{\hat{T}}{R}*\frac{\hat{T}}{R}\right.\\ &\quad\quad \left.+\frac{\hat{T}}{R}*\frac{\hat{T}}{R}*\frac{\delta\hat{T}}{R}*\frac{\hat{T}}{R}+\frac{\hat{T}}{R}*\frac{\hat{T}}{R}*\frac{\hat{T}}{R}*\frac{\delta\hat{T}}{R}\right)+\cdots\\ &= -\left[\frac{\sinh(\hat{T}/R)\delta\hat{T}}{R\cosh^2(\hat{T}/R)}\right]_*, \end{aligned} \quad (A.26)$$

Substituting Eq. (A.26) into Eq. (A.24) and making use of the cyclic property of star product, we rewrite Eq. (A.24) as

$$\delta_{\hat{T}}\hat{S}_1[\hat{T}] = \mathcal{T}_p \int d^{p+1}x \left[\frac{\sinh(\hat{T}/R)}{R\cosh^2(\hat{T}/R)} \right]_* \sqrt{-\hat{X}}\,\delta\hat{T}. \tag{A.27}$$

Inserting the formula of cofactor (2.27) into Eq. (A.25), we arrive at

$$\begin{aligned} \delta_{\hat{T}}\hat{S}_2[\hat{T}] &= \frac{\hat{\mathcal{T}}_p}{2} \int d^{p+1}x \,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}} \hat{C}^{\mu\nu}\,\delta_{\hat{T}}\hat{X}_{\mu\nu}, \\ &= \hat{\mathcal{T}}_p \int d^{p+1}x \,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}_{\rm S}^{\mu\nu}\,\hat{D}_\mu\delta\hat{T}\hat{D}_\nu\hat{T} \\ &= -\hat{\mathcal{T}}_p \int d^{p+1}x\,\hat{D}_\mu \left(\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}_{\rm S}^{\mu\nu}\,\hat{D}_\nu\hat{T} \right) \delta\hat{T}. \end{aligned} \tag{A.28}$$

In the last step of Eq. (A.28) we have used the following property of the star product

$$\int \hat{f} * \hat{D}_\mu\hat{g} = \int \partial_\mu(\hat{f} * \hat{g}) - \int (\hat{D}_\mu\hat{f}) * \hat{g} = -\int (\hat{D}_\mu\hat{f})\hat{g}. \tag{A.29}$$

Plugging Eqs. (A.27)–(A.28) into the variation of the action (A.23), we find the equation of motion for the NC tachyon field (2.25).

B.2 NC U(1) gauge field equation

Similar to the previous section, we calculate the variation of NC action under a small variation of the NC gauge field $\hat{A}_\mu$

$$\begin{aligned} \delta_{\hat{A}}\hat{S}[\hat{A}] &= \hat{S}[\hat{A}+\delta\hat{A}] - \hat{S}[\hat{A}] = \int d^{p+1}x\,\frac{\delta\hat{S}[\hat{A}]}{\delta\hat{A}_\mu(x)}\delta\hat{A}_\mu(x) \\ &= -\hat{\mathcal{T}}_p \int d^{p+1}x\,\hat{V}\,\delta_{\hat{A}}\sqrt{-\hat{X}} \end{aligned} \tag{A.30}$$

$$\begin{aligned} &= \frac{\mathcal{T}_p}{2} \int d^{p+1}x\,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}^{\mu\nu}\delta_{\hat{A}}\hat{X}_{\mu\nu} \\ &= \delta_{\hat{A}}\hat{S}_1[\hat{A}] + \delta_{\hat{A}}\hat{S}_2[\hat{A}], \end{aligned} \tag{A.31}$$

where

$$\delta_{\hat{A}}\hat{S}_1[\hat{A}] = \frac{\mathcal{T}_p}{2} \int d^{p+1}x\,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}_{\rm A}^{\mu\nu}\delta_{\hat{A}}\hat{F}_{\mu\nu}, \tag{A.32}$$

$$\delta_{\hat{A}}\hat{S}_2[\hat{A}] = \mathcal{T}_p \int d^{p+1}x\,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}_{\rm S}^{\mu\nu}\,\delta_{\hat{A}}\left(\hat{D}_\mu\hat{T}\right)\hat{D}_\nu\hat{T}. \tag{A.33}$$

From the definition of NC gauge field strength (2.7) we obtain

$$\begin{aligned}\delta_{\hat{A}}\hat{S}_1[\hat{A}] &= \mathcal{T}_p\int d^{p+1}x\,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}^{\mu\nu}_{\rm A}\hat{D}_\mu\delta\hat{A}_\nu \\ &= -\mathcal{T}_p\int d^{p+1}x\,\hat{D}_\mu\left(\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}^{\mu\nu}_{\rm A}\right)\delta\hat{A}_\nu,\end{aligned} \tag{A.34}$$

and the definition of the covariant derivative (2.10) leads to

$$\begin{aligned}\delta_{\hat{A}}\hat{S}_2[\hat{A}] &= -i\hat{\mathcal{T}}_p\int d^{p+1}x\,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}^{\mu\nu}_{\rm S}\hat{D}_\nu\hat{T}*\left(\delta\hat{A}_\mu*\hat{T}-\hat{T}*\delta\hat{A}_\mu\right) \\ &= -i\hat{\mathcal{T}}_p\int d^{p+1}x\,\left[\hat{T}\,,\,\frac{\hat{V}(\hat{T})}{\sqrt{-\hat{X}}}\,\hat{C}^{\mu\nu}_{\rm S}\hat{D}_\nu\hat{T}\right]_*\delta\hat{A}_\mu,\end{aligned} \tag{A.35}$$

where we used

$$\delta_{\hat{A}}(\hat{D}_\mu\hat{T}) = -i(\delta\hat{A}_\mu*\hat{T}-\hat{T}*\delta\hat{A}_\mu). \tag{A.36}$$

Substituting Eqs. (A.34)–(A.35) into Eq. (A.31) provides the gauge field equation (2.26).

References

[1] E. S. Fradkin and A. A. Tseytlin, "Nonlinear Electrodynamics From Quantized Strings," Phys. Lett. B **163**, 123 (1985);

C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, "String Loop Corrections To Beta Functions," Nucl. Phys. B **288**, 525 (1987);

A. Abouelsaood, C. G. Callan, C. R. Nappi and S. A. Yost, "Open Strings In Background Gauge Fields," Nucl. Phys. B **280**, 599 (1987).

[2] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].

[3] N. Seiberg, L. Susskind and N. Toumbas, "Space/time non-commutativity and causality," JHEP **0006**, 044 (2000) [arXiv:hep-th/0005015];

J. L. F. Barbon and E. Rabinovici, "Stringy fuzziness as the custodian of time-space noncommutativity," Phys. Lett. B **486**, 202 (2000) [arXiv:hep-th/0005073].

[4] R. Gopakumar, J. M. Maldacena, S. Minwalla and A. Strominger, "S-duality and noncommutative gauge theory," JHEP **0006**, 036 (2000) [arXiv:hep-th/0005048];

N. Seiberg, L. Susskind and N. Toumbas, "Strings in background electric field, space/time noncommutativity and a new noncritical string theory," JHEP **0006**, 021 (2000) [arXiv:hep-th/0005040];

S. J. Rey and R. von Unge, "S-duality, noncritical open string and noncommutative gauge theory," Phys. Lett. B **499**, 215 (2001) [arXiv:hep-th/0007089].

[5] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, "OM theory in diverse dimensions," JHEP **0008**, 008 (2000) [arXiv:hep-th/0006062].

[6] R. Gopakumar, S. Minwalla and A. Strominger, "Noncommutative solitons," JHEP **0005**, 020 (2000) [arXiv:hep-th/0003160].

[7] K. Dasgupta, S. Mukhi and G. Rajesh, "Noncommutative tachyons," JHEP **0006**, 022 (2000) [arXiv:hep-th/0005006].

[8] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, "D-branes and strings as non-commutative solitons," JHEP **0007**, 042 (2000) [arXiv:hep-th/0005031].

[9] A. P. Polychronakos, "Flux tube solutions in noncommutative gauge theories," Phys. Lett. B **495**, 407 (2000) [arXiv:hep-th/0007043];

D. P. Jatkar, G. Mandal and S. R. Wadia, "Nielsen-Olesen vortices in noncommutative Abelian Higgs model," JHEP **0009**, 018 (2000) [arXiv:hep-th/0007078];

D. Bak, "Exact multi-vortex solutions in noncommutative Abelian-Higgs theory," Phys. Lett. B **495**, 251 (2000) [arXiv:hep-th/0008204];

D. Bak, K. M. Lee and J. H. Park, "Noncommutative vortex solitons," Phys. Rev. D **63**, 125010 (2001) [arXiv:hep-th/0011099];

G. S. Lozano, E. F. Moreno and F. A. Schaposnik, "Nielsen-Olesen vortices in noncommutative space," Phys. Lett. B **504**, 117 (2001) [arXiv:hep-th/0011205];

G. S. Lozano, E. F. Moreno and F. A. Schaposnik, "Self-dual Chern-Simons solitons in noncommutative space," JHEP **0102**, 036 (2001) [arXiv:hep-th/0012266].

[10] D. Ghoshal, "Exact noncommutative solitons in p-adic strings and BSFT," arXiv:hep-th/0406259.

[11] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, "Unstable solitons in noncommutative gauge theory," JHEP **0104**, 001 (2001) [arXiv:hep-th/0009142];

P. Kraus, A. Rajaraman and S. H. Shenker, "Tachyon condensation in noncommutative gauge theory," Nucl. Phys. B **598**, 169 (2001) [arXiv:hep-th/0010016];

G. Mandal and S. R. Wadia, "Matrix model, noncommutative gauge theory and the tachyon potential," Nucl. Phys. B **599**, 137 (2001) [arXiv:hep-th/0011094].

[12] R. Gopakumar, S. Minwalla and A. Strominger, "Symmetry restoration and tachyon condensation in open string theory," JHEP **0104**, 018 (2001) [arXiv:hep-th/0007226];

N. Seiberg, "A note on background independence in noncommutative gauge theories, matrix model and tachyon condensation," JHEP **0009**, 003 (2000) [arXiv:hep-th/0008013];

Y. Hikida, M. Nozaki and T. Takayanagi, "Tachyon condensation on fuzzy sphere and noncommutative solitons," Nucl. Phys. B **595**, 319 (2001) [arXiv:hep-th/0008023];

J. A. Harvey and G. W. Moore, "Noncommutative tachyons and K-theory," J. Math. Phys. **42**, 2765 (2001) [arXiv:hep-th/0009030];

A. Sen, "Some issues in non-commutative tachyon condensation," JHEP **0011**, 035 (2000) [arXiv:hep-th/0009038];

M. Li, "Note on noncommutative tachyon in matrix models," Nucl. Phys. B **602**, 201 (2001) [arXiv:hep-th/0010058].

[13] A. Sen, "Tachyon condensation on the brane antibrane system," JHEP **9808**, 012 (1998) [arXiv:hep-th/9805170];

For reviews see A. Sen, "Non-BPS states and branes in string theory," arXiv:hep-th/9904207;

A. Lerda and R. Russo, "Stable non-BPS states in string theory: A pedagogical review," Int. J. Mod. Phys. A **15**, 771 (2000) [arXiv:hep-th/9905006];

J. H. Schwarz, "TASI lectures on non-BPS D-brane systems," arXiv:hep-th/9908144.

[14] A. Sen, "Dirac-Born-Infeld action on the tachyon kink and vortex," Phys. Rev. D **68**, 066008 (2003) [arXiv:hep-th/0303057].

[15] N. Lambert, H. Liu and J. Maldacena, "Closed strings from decaying D-branes," arXiv:hep-th/0303139.

[16] C. Kim, Y. Kim and C. O. Lee, "Tachyon kinks," JHEP **0305**, 020 (2003) [arXiv:hep-th/0304180].

[17] P. Brax, J. Mourad and D. A. Steer, "Tachyon kinks on non BPS D-branes," Phys. Lett. B **575**, 115 (2003) [arXiv:hep-th/0304197].

[18] C. Kim, Y. Kim, O-K. Kwon and C. O. Lee, "Tachyon kinks on unstable Dp-branes," JHEP **0311**, 034 (2003) [arXiv:hep-th/0305092].

[19] G. Mandal and S. J. Rey, "A note on D-branes of odd codimensions from noncommutative tachyons," Phys. Lett. B **495**, 193 (2000) [arXiv:hep-th/0008214].

[20] M. R. Garousi, "Tachyon couplings on non-BPS D-branes and Dirac-Born-Infeld action," Nucl. Phys. B **584**, 284 (2000) [arXiv:hep-th/0003122];

E. A. Bergshoeff, M. de Roo, T. C. de Wit, E. Eyras and S. Panda, "T-duality and actions for non-BPS D-branes," JHEP **0005**, 009 (2000) [arXiv:hep-th/0003221];

J. Kluson, "Proposal for non-BPS D-brane action," Phys. Rev. D **62**, 126003 (2000) [arXiv:hep-th/0004106].

[21] A. A. Gerasimov and S. L. Shatashvili, "On exact tachyon potential in open string field theory," JHEP **0010**, 034 (2000) [arXiv:hep-th/0009103];

D. Kutasov, M. Marino and G. W. Moore, "Some exact results on tachyon condensation in string field theory," JHEP **0010**, 045 (2000) [arXiv:hep-th/0009148];

D. Kutasov, M. Marino and G. W. Moore, "Remarks on tachyon condensation in superstring field theory," arXiv:hep-th/0010108.

[22] A. Sen, "Open and closed strings from unstable D-branes," Phys. Rev. D **68**, 106003 (2003) [arXiv:hep-th/0305011].

[23] P. Mukhopadhyay and A. Sen, "Decay of unstable D-branes with electric field," JHEP **0211**, 047 (2002) [arXiv:hep-th/0208142].

[24] R. Banerjee, C. Lee and H. S. Yang, "Seiberg-Witten-type maps for currents and energy-momentum tensors in noncommutative gauge theories," arXiv:hep-th/0312103;

R. Banerjee and K. Kumar, "Maps for currents and anomalies in noncommutative gauge theories: classical and quantum aspects," arXiv:hep-th/0404110.

[25] A. Sen, "Universality of the tachyon potential," JHEP **9912**, 027 (1999) [arXiv:hep-th/9911116].

[26] A. Sen, “Field theory of tachyon matter,” Mod. Phys. Lett. A **17**, 1797 (2002) [arXiv:hep-th/0204143].

[27] A. Buchel, P. Langfelder and J. Walcher, “Does the tachyon matter?,” Annals Phys. **302**, 78 (2002) [arXiv:hep-th/0207235];

[28] C. Kim, H. B. Kim, Y. Kim and O-K. Kwon, “Electromagnetic string fluid in rolling tachyon,” JHEP **0303**, 008 (2003) [arXiv:hep-th/0301076];

C. Kim, H. B. Kim, Y. Kim and O-K. Kwon, “Cosmology of rolling tachyon,” proceedings of PAC Memorial Symposium on Theoretical Physics, pp.209-239 (Chungbum Publishing House, Seoul, 2003), arXiv:hep-th/0301142;

[29] F. Leblond and A. W. Peet, “SD-brane gravity fields and rolling tachyons,” JHEP **0304**, 048 (2003) [arXiv:hep-th/0303035].

[30] D. Kutasov and V. Niarchos, “Tachyon effective actions in open string theory,” Nucl. Phys. B **666**, 56 (2003) [arXiv:hep-th/0304045].

[31] See, for example, S. Stahl, “The Poincaré Half-Plane — A Gatway to Modern Geometry” (Jones and Bartlett Publishers, Boston, 1993), Chapter 16.

[32] C. Kim, Y. Kim, O-K. Kwon and P. Yi, “Tachyon tube and supertube,” JHEP **0309**, 042 (2003) [arXiv:hep-th/0307184].

[33] C. Kim, Y. Kim and O. K. Kwon, “Tubular D-branes in Salam-Sezgin model,” JHEP **0405**, 020 (2004) [arXiv:hep-th/0404163].

[34] A. Sen, “Moduli space of unstable D-branes on a circle of critical radius,” JHEP **0403**, 070 (2004) [arXiv:hep-th/0312003].

[35] I. Y. Aref’eva, D. M. Belov and A. S. Koshelev, “A note on UV/IR for noncommutative complex scalar field,” arXiv:hep-th/0001215.

[36] Y. Kiem, C. Kim and Y. Kim, “Noncommutative Q-balls,” Phys. Lett. B **507**, 207 (2001) [arXiv:hep-th/0102160].

[37] L.D. Landau and E.M. Lifshitz, section 12 of *Mechanics*, 3rd ed, (Pergamon press, 1976)

[38] O-K. Kwon and P. Yi, “String fluid, tachyon matter, and domain walls,” JHEP **0309**, 003 (2003) [arXiv:hep-th/0305229].

[39] E. Witten, "Noncommutative tachyons and string field theory," arXiv:hep-th/0006071;

L. Cornalba, "Tachyon condensation in large magnetic fields with background independent string field theory," Phys. Lett. B **504**, 55 (2001) [arXiv:hep-th/0010021];

K. Okuyama, "Noncommutative tachyon from background independent open string field theory," Phys. Lett. B **499**, 167 (2001) [arXiv:hep-th/0010028].

[40] K. Hashimoto and T. Hirayama, "Branes and BPS configurations of noncommutative / commutative gauge theories," Nucl. Phys. B **587**, 207 (2000) [arXiv:hep-th/0002090];

J. L. Karczmarek and C. G. Callan, "Tilting the noncommutative bion," JHEP **0205**, 038 (2002) [arXiv:hep-th/0111133].

[41] V. O. Rivelles, "Noncommutative field theories and gravity," Phys. Lett. B **558**, 191 (2003) [arXiv:hep-th/0212262].

[42] H. S. Yang, "Exact Seiberg-Witten Map and Induced Gravity from Noncommutativity," arXiv:hep-th/0402002;

R. Banerjee and H. S. Yang, "Exact Seiberg-Witten map, induced gravity and topological invariants in noncommutative field theories," arXiv:hep-th/0404064.